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# Parallel Transport in Gauge Theory on $M_4 \times Z_2$ Geometry

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## Abstract

We apply the gauge theory on  $M_4 \times Z_2$  geometry previously proposed by Konisi and Saito to the Weinberg-Salam model for electroweak interactions, especially in order to clarify the geometrical meaning of curvatures in this geometry. Considering the Higgs field to be a gauge field along  $Z_2$  direction, we also discuss the BRST invariant gauge fixing in this theory.

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## §1. Introduction

The noncommutative geometry(NCG) of Connes[1, 2] has been successful in giving a geometrical interpretation of the standard model as well as some grand unification models. In this interpretation the Higgs fields are regarded as gauge fields along directions in the discrete space. The bosonic parts of actions are just the pure Yang-Mills actions containing gauge fields on both continuous and discrete spaces, and the Yukawa coupling is regarded as a kind of gauge interactions of fermions.

There are now various alternative versions of NCG[3]. Any NCG, however, has so far been algebraic rather than geometric. Nobody has considered enough the original geometric meaning such as covariant differences, parallel transportations, curvature and so on in the discrete space. In previous works, one of the authors (T.S.) collaborated with Konisi has considered such a geometric meaning of NCG and proposed the gauge theory on  $M_4 \times Z_N$  without recourse to any knowledge of NCG, where  $M_4$  is the four-dimensional Minkowski space and  $Z_N$  is the discrete space with  $N$  points. Here the Higgs fields have been introduced as mapping functions between any pair of vector fields belonging independently to the  $N$ -sheeted space-time, just as the Yang-Mills field is so between both vectors on  $x$  and  $x + \delta x$ . We have applied this gauge theory to the Weinberg-Salam(WS) model for electroweak interactions,  $N = 2$  and 4 super Yang-Mills theories and the Brans-Dicke theory for gravity[4].

In the present paper we revisit the above gauge theory on  $M_4 \times Z_N$  geometry, especially for the  $Z_2$  case, because this has not so far been discussed enough. The WS model can be interpreted as the gauge theory on  $M_4 \times Z_2$  geometry, where the Higgs field is the gauge field associated with  $Z_2$ . In this geometry one can consider two kinds of curvatures for the Higgs field. Our purpose is to clarify the geometrical meaning of these curvatures. We also discuss the BRST invariant gauge fixing in this gauge theory. At first sight one may wonder whether the Higgs field requires a new ghost, because it is a gauge field. This question becomes clear and is eventually reduced to the conventional  $R_\xi$ -gauge fixing plus Faddeev-Popov(FP) ghosts for the WS model with spontaneously broken symmetry of  $SU(2) \times U(1)$ . There are now two similar works in this gauge fixing[5]. Comparing with their works our approach which does not use NCG seems to be much simpler and clearer.

In §2 we consider the gauge theory of the WS model on  $M_4 \times Z_2$ , especially clarifying the geometrical meaning of curvatures for the Higgs field. In §3 we discuss the BRST invariant gauge fixing plus FP ghosts in this geometry. The final section is devoted to concluding remarks.

## §2. The Weinberg-Salam model on $M_4 \times Z_2$ geometry

To every point  $(x, p)$  with  $x \in M_4$  and  $p \in Z_2$  we attach a complex  $n_p$ -dimensional internal vector space  $V[n_p, x, p]$ . Let us take

$$\begin{aligned} V[2, x, +] &= \text{complex 2 dimensional space,} \\ V[1, x, -] &= \text{complex 1 dimensional space,} \\ p &= (+, -) \in Z_2. \end{aligned} \quad (2.1)$$

The fermionic fields  $\psi(x, p)$  are chosen as

$$\psi(x, +) = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \text{and} \quad \psi(x, -) = e_R, \quad (2.2)$$

where the left-handed neutrino  $\nu_L$  and electron  $e_L$  are the  $SU(2)$  doublet and the right-handed electron  $e_R$  is the  $SU(2)$  singlet. Since  $\nu_L$  and  $e_L$  have the hypercharge  $Y = -1$  and  $e_R$  has  $Y = -2$ , the gauge field  $\omega_\mu(x, p)$  coupled to  $\psi(x, +)$  and  $\psi(x, -)$  should be introduced as

$$(\omega_\mu(x, +))_j^i = \frac{1}{2}[g'\tau^0 B_\mu(x) - g\tau^a A_\mu^a(x)]_j^i, \quad (i, j = 1, 2) \quad (2.3)$$

$$(\omega_\mu(x, -))_0^0 = g'B_\mu(x), \quad (2.4)$$

respectively, where  $\tau^a (a = 1, 2, 3)$  is the Pauli matrix and  $\tau^0$  is the  $2 \times 2$  unit matrix. The field strengths (or curvatures) for  $\omega_\mu(x, \pm)$  are, therefore, given by

$$F_{\mu\nu}(+) = \frac{1}{2}(g'\tau^0 B_{\mu\nu} - g\tau^a F_{\mu\nu}^a), \quad (2.5)$$

$$F_{\mu\nu}(-) = g'B_{\mu\nu}, \quad (2.6)$$

with

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (2.7)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon_{abc}A_\mu^b A_\nu^c. \quad (2.8)$$

Let  $\psi^i(x, p)$  be any vector field on  $V[n_p, x, p]$ . We define covariant differences along the  $Z_2$ -direction by

$$\delta_H \psi^i(x, +) = \psi^i(x, +) - H_0^i(x, +, -)\psi^0(x, -), \quad (2.9)$$

$$\delta_H \psi^0(x, -) = \psi^0(x, -) - H_i^0(x, -, +)\psi^i(x, +), \quad (i = 1, 2) \quad (2.10)$$

where  $H_0^i(x, +, -)$  is the mapping function of  $\psi^0(x, -)$  from  $(x, -)$  to  $(x, +)$  and  $H_i^0(x, -, +)$  is that of  $\psi^i(x, +)$  from  $(x, +)$  to  $(x, -)$ . They are subject to transformation rules

$$H_0^i(x, +, -) \rightarrow \widetilde{H}_0^i(x, +, -) = U_j^i(x, +)H_0^j(x, +, -)(U^{-1}(x, -))_0^0, \quad (2.11)$$

$$H_i^0(x, -, +) \rightarrow \widetilde{H}_i^0(x, -, +) = U_0^i(x, -)H_i^0(x, -, +)(U^{-1}(x, +))_i^j, \quad (2.12)$$

under gauge transformations

$$\psi^i(x, +) \rightarrow \tilde{\psi}^i(x, +) = U_j^i(x, +)\psi^j(x, +), \quad (2.13)$$

$$\psi^0(x, -) \rightarrow \tilde{\psi}^0(x, -) = U_0^0(x, -)\psi^0(x, -). \quad (2.14)$$

where  $U(x, \pm)$  are parametrized as

$$U(x, \pm) = \exp\{i\theta(x, \pm)\} \quad (2.15)$$

with

$$\theta(x, +) = \frac{g'}{2}\tau^0\theta(x) - \frac{g}{2}\tau^a\theta^a(x), \quad \theta(x, -) = g'\theta(x). \quad (2.16)$$

Namely,  $U(x, +)$  and  $U(x, -)$  stand for local rotations of  $SU(2) \times U_Y(1)$  and  $U_Y(1)$ , respectively. The rules (2.11)–(2.14) guarantee vector properties of  $\delta_H\psi^i(x, +)$  and  $\delta_H\psi^0(x, -)$  on  $V[n_p, x, p]$ . The situation is quite the same as before for gauge fields  $\omega_\mu(x, p)$ , where the mapping functions are given by

$$H(x + \delta x, x, p) = 1 + i\omega_\mu(x, p)\delta x^\mu + \dots \quad (2.17)$$

In this case a covariant variation of  $\psi^i(x, p)$  due to an infinitesimal displacement  $\delta x^\mu$  is given by

$$\delta\psi^i(x, p) = \psi^i(x + \delta x, p) - \psi_{\parallel}^i(x + \delta x, p), \quad (2.18)$$

where  $\psi_{\parallel}^i(x + \delta x, p)$  is a parallel-transported vector of  $\psi^i(x, p)$  from  $x$  to  $x + \delta x$ , *i.e.*,

$$\psi_{\parallel}^i(x + \delta x, p) = H_j^i(x + \delta x, x, p)\psi^j(x, p). \quad (2.19)$$

Here, the mapping function  $H_j^i$  should be subject to the transformation rule under the rotation  $U(x, p)$

$$H(x + \delta x, x, p) \rightarrow \widetilde{H}(x + \delta x, x, p) = U(x + \delta x, p)H(x + \delta x, x, p)U^{-1}(x, p). \quad (2.20)$$

This rule guarantees the vector property of  $\psi_{\parallel}(x + \delta x, p)$  on  $V[n_p, x + \delta x, p]$ . If we use the familiar notations (2.17) and

$$U(x + \delta x, p) = U(x, p) + \partial_{\mu}U(x, p)\delta x^{\mu} + \dots \quad (2.21)$$

and keep terms up to the first-order  $\delta x^{\mu}$  in Eq.(2.20), we have the gauge transformation rule for the non-Abelian gauge field  $\omega_{\mu}(x, p)$

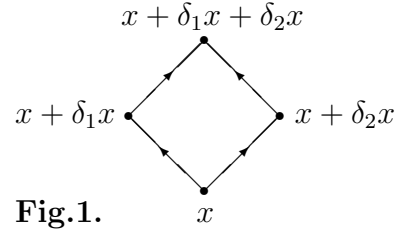
$$\omega_{\mu}(x, p) \rightarrow \tilde{\omega}_{\mu}(x, p) = U(x, p) \omega_{\mu}(x, p) U^{-1}(x, p) - i \partial_{\mu}U(x, p) U^{-1}(x, p). \quad (2.22)$$

Comparing (2.9) – (2.12) with (2.18) – (2.20) we find that the mapping function  $H(x, p, p')$  can be regarded as gauge fields associated with  $Z_2$ . Note that the gauge transformation rules (2.11) and (2.12) cannot be reduced to the form (2.22) with inhomogeneous terms, because  $H_0^i(x, +, -)$  and  $H_i^0(x, -, +)$  do not have Kronecker delta terms. In order to guarantee the hermiticity of the Yukawa coupling terms with fermions which are nothing but gauge interactions, we assume

$$H_0^i(x, +, -) = (H_i^0(x, -, +))^* = H^i(x) = (H^+, H^0), \quad (2.23)$$

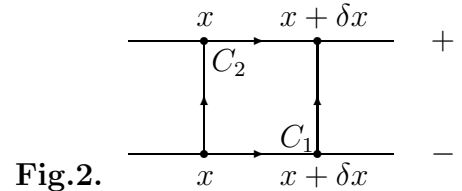
where  $*$  denotes the complex conjugation.

We now consider field strengths or curvatures for the gauge field  $H(x, p, p')$ . Let us call the usual field strength  $F_{\mu\nu}(x, p)$  for  $\omega_{\mu}(x, p)$  the curvature of the first type. As well known, this comes from a difference between two parallel transportations of  $\psi^i(x, p)$  along two paths depicted in Fig.1.



**Fig.1.**

In quite the same way we consider a curvature  $F_{\mu H}(x, +)$  of the second type which is defined by a difference between two mappings of  $\psi^0(x, -)$  along paths  $C_1$  and  $C_2$  depicted in Fig.2. The two mappings are given by



**Fig.2.**

$$C_1 = H_0^i(x + \delta x, +, -)H_0^0(x + \delta x, x, -)\psi^0(x, -), \quad (2.24)$$

$$C_2 = H_j^i(x + \delta x, x, +)H_0^j(x, +, -)\psi^0(x, -). \quad (2.25)$$

Substituting (2.17) and (2.23) into  $C_1$  and  $C_2$  above, we have

$$\begin{aligned} C_1 - C_2 &= [\partial_\mu H(+, -) + i\omega_\mu(+ )H(+, -) - iH(+, -)\omega_\mu(-)]\psi^0(-)\delta x^\mu \\ &= [\partial_\mu H(x) - \frac{i}{2}(g'\tau^0 B_\mu + g\tau^a A_\mu^a)H(x)]\psi^0(-)\delta x^\mu \\ &\equiv (\nabla_\mu H(x))\psi^0(-)\delta x^\mu. \end{aligned} \quad (2.26)$$

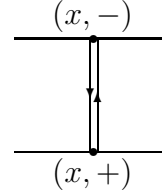
The second type curvature components  $F_{\mu H}(x, +)$  are, therefore, given by

$$(F_{\mu H}(x, +))_0^i = (\nabla_\mu H(x))^i. \quad (2.27)$$

In the same way we have

$$(F_{\mu H}(x, -))_i^0 = (\nabla_\mu H(x))_i^\dagger. \quad (2.28)$$

A curvature of the third type  $F_{HH}(x, +)$  corresponds to Fig.3. Namely,  $\psi^i(x, +)$  is compared with  $\psi_\parallel^i(x, +)$  which is the mapped function of  $\psi^i(x, +)$  from  $(x, +)$  to  $(x, -)$  and then returning to  $(x, +)$ , *i.e.*,



**Fig.3.**

$$\begin{aligned} \psi^i(x, +) - \psi_\parallel^i(x, +) &= \psi^i(x, +) - H_0^i(x, +, -)H_j^0(x, -, +)\psi^j(x, +) \\ &= (\delta_j^i - H^i(x)H_j^\dagger(x))\psi^j(x, +). \end{aligned} \quad (2.29)$$

This gives the third type curvature

$$(F_{HH}(x, +))_j^i = \delta_j^i - H^i(x)H_j^\dagger(x). \quad (2.30)$$

In the same way we have

$$(F_{HH}(x, -))_0^i = 1 - H_i^\dagger(x)H^i(x). \quad (2.31)$$

On  $M_4$  we know that there is no curvature of the similar type corresponding to two paths:  $x \rightarrow x + \delta x \rightarrow x$  and  $x \rightarrow x$ . On  $Z_2$ , however, we find non-vanishing curvature of the third type (see Appendix).

Now, considering  $A_\mu^a(x)$ ,  $B_\mu(x)$  and  $H^i(x)$  to be gauge fields, a Lagrangian for them should be of the Yang-Mills type

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \quad (2.32)$$

where

$$\begin{aligned} \mathcal{L}_1 &= -\sum_p \frac{1}{4c_p^2} \text{tr}[F_{\mu\nu}^\dagger(p) F^{\mu\nu}(p)] \\ &= -\frac{1}{8c_+^2} (g^2 F_{\mu\nu}^a F^{a\mu\nu} + g'^2 B_{\mu\nu} B^{\mu\nu}) - \frac{1}{4c_-^2} g'^2 B_{\mu\nu} B^{\mu\nu} \\ &= -\frac{g^2}{8c_+^2} F_{\mu\nu}^a F^{a\mu\nu} - \left(\frac{g'^2}{8c_+^2} + \frac{g'^2}{4c_-^2}\right) B_{\mu\nu} B^{\mu\nu}, \end{aligned} \quad (2.32a)$$

$$\begin{aligned} \mathcal{L}_2 &= -\sum_p \xi_p \text{tr}[F_{\mu H}^\dagger(p) F^{\mu H}(p)] \\ &= -(\xi_+ + \xi_-)(\nabla_\mu H)^\dagger (\nabla^\mu H), \end{aligned} \quad (2.32b)$$

$$\begin{aligned} \mathcal{L}_3 &= -\sum_p \zeta_p \text{tr}[F_{HH}^\dagger(p) F^{HH}(p)] \\ &= -\zeta_+ \text{tr}(1 - HH^\dagger)(1 - HH^\dagger) - \zeta_- (1 - H^\dagger H)^2 \\ &= -(\zeta_+ + \zeta_-) \left[ (H^\dagger H - 1)^2 + \frac{\zeta_+}{\zeta_+ + \zeta_-} \right], \end{aligned} \quad (2.32c)$$

where  $c_p$ ,  $\xi_p$  and  $\zeta_p$  are normalization constants. They should be so chosen as to be consistent with positivity of kinetic terms and renormalizability of the theory. Let us normalize  $\mathcal{L}_1$  by

$$\mathcal{L}_1 = -\frac{1}{4} (F_{\mu\nu}^a F^{a\mu\nu} + B_{\mu\nu} B^{\mu\nu}), \quad (2.33)$$

so that

$$g = \sqrt{2}c_+, \quad g' = \frac{\sqrt{2}c_+c_-}{\sqrt{2c_+^2 + c_-^2}}. \quad (2.34)$$

This means that  $g$  and  $g'$  are still independent parameters with each other. If we redefine the scalar field  $H$  by

$$\phi = \sqrt{-\xi_+ - \xi_-} H, \quad (2.35)$$

$\mathcal{L}_2$  and  $\mathcal{L}_3$  are reduced to the original WS type

$$\mathcal{L}_2 = (\nabla_\mu \phi)^\dagger (\nabla^\mu \phi), \quad (2.36)$$

$$\mathcal{L}_3 = -\mu^2 |\phi|^2 - \lambda |\phi|^4 + \text{const.}, \quad (2.37)$$

where

$$\lambda = \frac{\zeta_+ + \zeta_-}{(\xi_+ + \xi_-)^2}, \quad \mu^2 = 2\frac{\zeta_+ + \zeta_-}{\xi_+ + \xi_-} = 2(\xi_+ + \xi_-)\lambda < 0. \quad (2.38)$$

Here we have assumed to be  $\xi_+ + \xi_- < 0$  and  $\zeta_+ + \zeta_- > 0$ . The  $\mathcal{L}_3$  is nothing but the Higgs potential which has a minimal value at  $H^\dagger H = 1$ , *i.e.*,  $|\phi|^2 = -(\xi_+ + \xi_-) = -\mu^2/2\lambda > 0$ . Both constants  $\lambda$  and  $\mu^2$  are still independent parameters with each other within  $\lambda > 0$  and  $\mu^2 < 0$ . The fermionic part will be neglected because it is irrelevant to our purpose, it can be seen in Ref.4.

### §3. Gauge fixing and FP ghosts

In §2 we have seen that the WS model can be interpreted as the gauge theory on  $M_4 \times Z_2$  geometry, where the Higgs field is the gauge field along the direction in the discrete space  $Z_2$ . The Higgs fields obeys the gauge transformation rule (2.11), *i.e.*,

$$H(+, -) \rightarrow \widetilde{H}(+, -) = U(+, -)H(+, -)U^{-1}(-). \quad (3.1)$$

For an infinitesimal gauge transformation  $U(\pm) \cong 1 + i\theta(\pm)$ , Eq.(3.1) becomes in the notations (2.35) and (2.16)

$$\begin{aligned} \delta\phi &= i\theta(+)\phi - i\phi\theta(-) \\ &= -\frac{i}{2}(g'\tau^0\theta + g\tau^a\theta^a)\phi. \end{aligned} \quad (3.2)$$

This shows that the  $\phi$  has the hypercharge  $Y = +1$  and coincides with the conventional Higgs scalar field. The BRST transformation  $\delta_B\phi$  is then obtained by replacing  $\theta$  and  $\theta^a$  by ghosts  $c^0$  and  $c^a$

$$\delta_B\phi = -\frac{i}{2}(g'\tau^0c^0 + g\tau^ac^a)\phi. \quad (3.3)$$

For other gauge fields  $B_\mu$  and  $A_\mu^a$  it follows from (2.22) that their BRST transformations are given by

$$\delta_B B_\mu = \partial_\mu c^0 \quad \text{and} \quad \delta_B A_\mu^a = \partial_\mu c^a + g\epsilon_{abc}A^b c^c \equiv D_\mu c^a. \quad (3.4)$$

Eq.(3.3) shows that the  $\phi$  does not require any new ghosts though it is a gauge field.

Thus we have seen that the gauge-fixing in our geometry is reduced to the conventional one. For completeness we give the full result of the BRST invariant  $R_\xi$ -gauge fixing plus



FP ghosts with spontaneously broken symmetry of  $SU(2) \times U_Y(1)$ . The BRST invariant Lagrangian for gauge fixing plus FP ghosts should be of the form[6]

$$\mathcal{L}_{\text{GF+FP}} = -i\delta_B(*), \quad (3.5)$$

where  $*$  is chosen as follows:

$$\begin{aligned} * &= \bar{c}^0(\partial^\mu B_\mu + \frac{1}{2}\alpha B^0 + \alpha M_W \chi^0) \\ &+ \bar{c}^a(\partial^\mu A_\mu^a + \frac{1}{2}\alpha B^a + \alpha M_W \chi^a). \end{aligned} \quad (3.6)$$

Here,  $\bar{c}^0$  and  $\bar{c}^a$  are anti-ghosts, Nakanishi-Lautrup fields  $B^0$  and  $B^a$  are defined by  $-i\delta_B \bar{c}^0 = B^0$  and  $-i\delta_B \bar{c}^a = B^a$ ,  $\alpha$  the gauge parameter, and  $M_W$  a mass parameter. The Higgs field  $\phi$  is parametrized as

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^2(x) + i\chi^1(x) \\ v + \psi(x) - i\chi^3(x) \end{pmatrix}, \quad (3.7)$$

where  $\psi(x)$ ,  $\chi^a(x)$ ,  $a = 1, 2, 3$ , are real scalar fields and  $v$  is a real constant. The  $\chi^a$  has been used in (3.6), while the  $\chi^0$  is the  $U_Y(1)$  phase factor of  $\phi$ .

Now,  $\mathcal{L}_{\text{GF+FP}}$  turns out to be of the form

$$\mathcal{L}_{\text{GF+FP}} = \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}, \quad (3.8)$$

where

$$\begin{aligned} \mathcal{L}_{\text{GF}} &= B^0(\partial^\mu B_\mu + \frac{1}{2}\alpha B^0 + \alpha M_W \chi^0) \\ &+ B^a(\partial^\mu A_\mu^a + \frac{1}{2}\alpha B^a + \alpha M_W \chi^a), \end{aligned} \quad (3.9)$$

$$\begin{aligned} \mathcal{L}_{\text{FP}} &= i\bar{c}^0(\partial^\mu \delta_B B_\mu + \alpha M_W \delta_B \chi^0) \\ &+ i\bar{c}^a(\partial^\mu \delta_B A_\mu^a + \alpha M_W \delta_B \chi^a). \end{aligned} \quad (3.10)$$

After rotating  $(B_\mu, A_\mu^3)$ ,  $(B^0, B^3)$  and  $(\chi^0, \chi^3)$  to  $(A_\mu, Z_\mu)$ ,  $(B_A, B_Z)$  and  $(\chi_A, \chi_Z)$  by the Weinberg angle  $\theta_W$ , respectively, *i.e.*,

$$\begin{aligned} A_\mu &= \cos \theta_W B_\mu + \sin \theta_W A_\mu^3, \quad B_A = \cos \theta_W B^0 + \sin \theta_W B^3, \\ Z_\mu &= -\sin \theta_W B_\mu + \cos \theta_W A_\mu^3, \quad B_Z = -\sin \theta_W B^0 + \cos \theta_W B^3, \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \chi_A &= \cos \theta_W \chi^0 + \sin \theta_W \chi^3, \\ \chi_Z &= -\sin \theta_W \chi^0 + \cos \theta_W \chi^3, \end{aligned} \quad (3.12)$$

we have

$$\begin{aligned}\mathcal{L}_{\text{GF}} = & B_Z(\partial^\mu Z_\mu + \frac{1}{2}\alpha B_Z + \alpha M_W \chi_Z) + B_A(\partial^\mu A_\mu + \frac{1}{2}\alpha B_A + \alpha M_W \chi_A) \\ & + B^1(\partial^\mu A_\mu^1 + \frac{1}{2}\alpha B^1 + \alpha M_W \chi^1) + B^2(\partial^\mu A_\mu^2 + \frac{1}{2}\alpha B^2 + \alpha M_W \chi^2).\end{aligned}\quad (3.13)$$

If we set  $\chi_A = 0$ , then it follows

$$\chi^0 = -\tan \theta_W \chi^3 = -\frac{g'}{g} \chi^3, \quad (3.14)$$

hence

$$\chi_Z = \frac{1}{\cos \theta_W} \chi^3 = \frac{M_Z}{M_W} \chi^3. \quad (3.15)$$

Thus, finally we obtain the  $R_\xi$ -gauge fixing Lagrangian[7]

$$\begin{aligned}\mathcal{L}_{\text{GF}} = & \frac{1}{2}\alpha(B_1^2 + B_2^2 + B_Z^2 + B_A^2) + B_1(\partial^\mu A_\mu^1 + \alpha M_W \chi^1) + B_2(\partial^\mu A_\mu^2 + \alpha M_W \chi^2) \\ & + B_Z(\partial^\mu Z_\mu + \alpha M_Z \chi^3) + B_A \partial^\mu A_\mu.\end{aligned}\quad (3.16)$$

Eliminating  $B$ -fields from  $\mathcal{L}_{\text{GF}}$ , it is reduced to

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\alpha} \sum_{a=1}^2 (\partial^\mu A_\mu^a + \alpha M_W \chi^a)^2 - \frac{1}{2\alpha} (\partial^\mu Z_\mu + \alpha M_Z \chi^3)^2 - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2. \quad (3.17)$$

In order to obtain the FP-ghost Lagrangian  $\mathcal{L}_{\text{FP}}$ , we need  $\delta_B \chi^a$  in (3.10). Substituting the parametrization (3.7) of  $\phi$  into (3.3), we have

$$\begin{aligned}\delta_B \chi^1 &= -\frac{g}{2}(\chi^2 c^3 - \chi^3 c^2) - \frac{g}{2}c^1(v + \psi) - \frac{g'}{2}c^0 \chi^2, \\ \delta_B \chi^2 &= -\frac{g}{2}(\chi^3 c^1 - \chi^1 c^3) - \frac{g}{2}c^2(v + \psi) + \frac{g'}{2}c^0 \chi^1, \\ \delta_B \chi^3 &= -\frac{g}{2}(\chi^1 c^2 - \chi^2 c^1) - \frac{g}{2}c^3(v + \psi) + \frac{g'}{2}c^0(v + \psi), \\ \delta_B \psi &= \frac{g}{2}(\chi^1 c^1 + \chi^2 c^2 + \chi^3 c^3) - \frac{g'}{2}c^0 \chi^3.\end{aligned}\quad (3.18)$$

By using the relation  $\chi^0 = -(g'/g)\chi^3$  and Eq.(3.4), the ghost Lagrangian  $\mathcal{L}_{\text{FP}}$  becomes

$$\begin{aligned}\mathcal{L}_{\text{FP}} = & -i\partial^\mu \bar{c}^0 \partial_\mu c^0 - i\partial^\mu \bar{c}^a D_\mu c^a - i\alpha M_W \frac{g}{2} [\bar{c}^a (\vec{\chi} \times \vec{c})^a + \bar{c}^a c^a (v + \psi)] \\ & - i\alpha M_W \frac{g'}{2} [\bar{c}^0 (c^1 \chi^2 - c^2 \chi^1) + (\bar{c}^1 \chi^2 - \bar{c}^2 \chi^1) c^0 - (\bar{c}^3 c^0 + \bar{c}^0 c^3)(v + \psi)] \\ & - i\alpha M_W \frac{g'}{2g} \bar{c}^0 c^0 (v + \psi).\end{aligned}\tag{3.19}$$

Thus we have obtained the BRST invariant  $R_\xi$ -gauge fixing and FP-ghost Lagrangians (3.16) and (3.19).

## §4. Concluding remarks

We have found that the covariant derivative of the Higgs field  $\nabla_\mu H$  is just the curvature of the second type corresponding to Fig.2 and the term  $1 - HH^\dagger$  is that of the third type corresponding to Fig.3. Since the Higgs field is one of gauge fields, its Lagrangian should be of Yang-Mills type. We have seen that this Lagrangian coincides exactly with that of the WS type with the Higgs potential.

We have also shown that the Higgs field does not require any new ghost though it is the gauge field. The BRST transformation of  $H$  coincides with the conventional one. Then we have given the full and new result of the BRST invariant  $R_\xi$ -gauge fixing plus FP ghosts for the WS model with spontaneous broken symmetry of  $SU(2) \times U_Y(1)$ .

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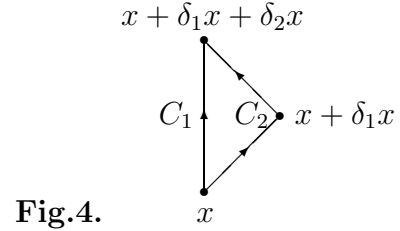
## Appendix

We would like to show that on  $M_4$  there is no curvature of the third type in the text. Let us consider two paths on  $M_4$

$$C_1 : x \rightarrow x + \delta_1 x \rightarrow (x + \delta_1 x) + \delta_2 x, \quad (\text{A.1})$$

$$C_2 : x \rightarrow x + (\delta_1 x + \delta_2 x). \quad (\text{A.2})$$

The difference  $\Delta = C_1 - C_2$  between two mapping functions is given by



$$\begin{aligned} \Delta &= C_1 - C_2 \\ &= H((x + \delta_1 x) + \delta_2 x, x + \delta_1 x) H(x + \delta_1 x, x) - H(x + (\delta_1 x + \delta_2 x), x). \end{aligned} \quad (\text{A.3})$$

Substituting the expression of  $H$

$$H(x + \delta x, x) = 1 + i \omega_\mu(x) \delta x^\mu + \frac{i}{2} C_{\mu\nu}(x) \delta x^\mu \delta x^\nu + \dots, \quad (\text{A.4})$$

where  $C_{\mu\nu}$  is symmetric with respect to  $\mu \nu$  interchanged, into (A.3), we have

$$\Delta = (-\omega_\mu \omega_\nu + i \omega_{\nu,\mu} - i C_{\mu\nu}) \delta_1 x^\mu \delta_2 x^\nu. \quad (\text{A.5})$$

If we choose  $\delta_2 x^\mu = \alpha \delta_1 x^\mu$  ( $\alpha > 0$ ), two paths  $C_1$  and  $C_2$  become the same. In this case the difference  $\Delta$  vanishes so that

$$C_{\mu\nu} = \frac{1}{2} (i \omega_{\{\mu} \omega_{\nu\}} - \omega_{\{\nu, \mu\}}). \quad (\text{A.6})$$

Substituting this into (A.5) we find

$$\begin{aligned} \Delta &= -\frac{i}{2} \{ \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + i [\omega_\mu, \omega_\nu] \} \delta_1 x^\mu \delta_2 x^\nu \\ &= -\frac{i}{2} F_{\mu\nu} \delta_1 x^\mu \delta_2 x^\nu, \end{aligned} \quad (\text{A.7})$$

which corresponds to one half of curvature for  $\omega_\mu$ .

Let us now choose  $\alpha = -1$ , *i.e.*,  $\delta_2 x^\mu = -\delta_1 x^\mu = -\delta x^\mu$ . Then Eq.(A.3) is reduced to

$$\begin{aligned}\Delta &= H(x, x + \delta x)H(x + \delta x, x) - 1 \\ &= \frac{i}{2}F_{\mu\nu}\delta x^\mu\delta x^\nu \\ &= 0,\end{aligned}\tag{A.8}$$

because  $F_{\mu\nu}$  is antisymmetric with respect to  $\mu$  and  $\nu$ . This shows that on  $M_4$  there is no curvature of the third type corresponding to two paths:  $x \rightarrow x + \delta x \rightarrow x$  and  $x \rightarrow x$ .

In the discrete space, say  $Z_N$ , there is no case such that two paths  $A \rightarrow B \rightarrow C$  and  $A \rightarrow C$  become the same. So we calculate directly  $\Delta$  in  $Z_2$  defined by

$$\Delta = 1 - H(+, -)H(-, +).\tag{A.9}$$

This is nothing but the curvature of the third type (see Fig.3).

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